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On some set-valued iteration semigroups generated by interval-valued functions

Grażyna Łydzińska

Abstract. Let X be an arbitrary set. We characterize all interval-valued functions $A : X \to 2^{\mathbb{R}}$ for which a multifunction $F : (0, \infty) \times X \to 2^X$ of the form $F(t, x) = A^{-}\big(A(x) + \min\{t, q - \inf A(x)\}\big)$, where $q = \sup A(X)$, is an iteration semigroup. The multifunction F is the set-valued counterpart of the fundamental form of continuous iteration semigroups of single-valued functions on an interval.

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Introduction

Let X be an arbitrary set X. A multifunction $F : (0, \infty) \times X \to 2^X$ is said to be a *set-valued iteration semigroup* if

$$F(s + t, x) = F\big(t, F(s, x)\big) \quad \text{for } x \in X \quad \text{and } s, t \in (0, \infty).$$

This notion was introduced and investigated by Smajdor [11] (see also e.g. [12]), studied by J. Olko (see e.g. [9,10]) and by Zdun [15]. In [4,5] we introduced and studied a family of set-valued functions which now will be denoted by (A) (see Sect. 1) and we showed (see [5, Remarks 1 and 3]) that F given by (A) is a set-valued counterpart of the fundamental form of iteration semigroups for single-valued functions which can be found in [2, Chap. IX, Sec. 1], [14, Theorems 5.1–8.1], [13, p. 98–99], [3, Chap. I, Sec. 1.7] (cf. also [1, Theorem 1]). In [7] we studied a lower semicontinuity of F given by (A).

In [8] we gave the necessary and sufficient conditions under which F given by (A) is an iteration semigroup (see [8, Theorems 2, 4 and 5]) (cf. also Sect. 1: Facts 3, 4 and 5).

In the present paper we will consider the case when the values of the generator A of the multifunction F are intervals. The main aim of this paper is to

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show all interval-valued functions A for which F given by (A) is an iteration semigroup.

1. Preliminaries

Fix a set X and a set-valued function $A : X \to 2^{\mathbb{R}}$ with non-empty values. Put

$$S := A(X) \quad \text{and} \quad q := \sup S.$$

Throughout this paper we will always assume that A satisfies the following condition:

(H) for every $s, t \in (0, \infty)$ and $x, z \in X$ with $\big[A(x) + s + t\big] \cap A(z) \neq \emptyset$ there exists $y \in X$ satisfying the conditions

$$[A(x) + s] \cap A(y) \neq \emptyset \tag{1}$$

and

$$[A(y) + t] \cap A(z) \neq \emptyset. \tag{2}$$

In the following remark we give some properties related to (H).

Remark 1. (i) If S is an interval then (H) holds (see [5, Proposition 1(i)]).

(ii) If at least one of the values of A is a singleton, then (H) holds if and only if S is an interval.

(iii) Assume that all values of A are intervals. If (H) holds then $(\inf S, \sup S) \subset \operatorname{cl} S$ (see [5, Proposition 1(iii)]).

(iv) Assume that all values of A are open sets. If $(\inf S, \sup S) \subset \operatorname{cl} S$ then (H) holds (see [5, Proposition 1(ii)]).

Proof. (ii) Assume that there exist $x \in X$ and $u \in S$ such that $A(x) = \{u\}$. By (i) it is enough to show that if (H) holds then S is an interval. Assume that (H) is satisfied and take $a, b \in S$, $a < b$. Let $c \in (a, b)$. We prove that $c \in S$. Take $w, z \in X$ such that $a \in A(w)$ and $b \in A(z)$. Of course if $c = u$ then $c \in S$, so assume that $c \neq u$.

If $u < c$ then there exist $s, t \in (0, \infty)$ such that

$$A(x) + s = \{c\} \quad \text{and} \quad A(x) + s + t = \{b\}.$$

Hence $[A(x) + s + t] \cap A(z) \neq \emptyset$. By (H) there exists $y \in X$ such that $[A(x) + s] \cap A(y) \neq \emptyset$, thus $c \in A(y) \subset S$.

Pass to the case $u > c$. We can find $s, t \in (0, \infty)$ such that

$$\{c\} + t = A(x) \quad \text{and} \quad \{a\} + s + t = A(x).$$

Thus $[A(w) + s + t] \cap A(x) \neq \emptyset$ and by (H) we get

$$A(y) \cap [A(x) - t] \neq \emptyset$$

for some $y \in X$. Since $A(x) - t = \{c\}$, we obtain that $c \in A(y)$. $\square$

For every $x \in X$ define

$$\tau(x) := q - \inf A(x).$$

Consider the following condition:

(H1) for every $x, z \in X$ and $s, t \in (0, \infty)$ with $s + t \leqslant \tau(x)$ if (1) and (2) hold for a $y \in X$ then $[A(x) + s + t] \cap A(z) \neq \emptyset$.

Notice that if A is single-valued then (H1) holds (see also [6, Remark 1]).

Remark 2. (see [8, Remark 2]) Assume that (H1) holds and $q = \infty$. Then for every $x \in X$ either card $A(x) = 1$ or diam $A(x) = \infty$.

Prove the following easy remark.

Remark 3. Assume that the condition (H1) holds. Let $x, z \in X$ and

$$u < w \quad \text{for} \;\; u \in A(x) \;\; \text{and} \;\; w \in A(z).$$

Then card $A(y) = 1$ for every $y \in Y$ such that

$$\sup A(x) \leqslant \inf A(y) \quad \text{and} \quad \sup A(y) \leqslant \inf A(z).$$

Proof. Take $y \in Y$ such that

$$\sup A(x) \leqslant \inf A(y) \quad \text{and} \quad \sup A(y) \leqslant \inf A(z)$$

and suppose that card $A(y) > 1$. Let $a, b \in A(y)$ and $a < b$. We can find $s \in (0, \infty)$ satisfying the conditions

$$\sup A(x) + s < \frac{a+b}{2} \tag{3}$$

and

$$a \in A(x) + s. \tag{4}$$

Similarly there exists $t \in (0, \infty)$ such that

$$\frac{a+b}{2} < \inf A(z) - t \tag{5}$$

and

$$b \in A(z) - t. \tag{6}$$

Observe that, by (3) and (5), we have

$$s + t < \inf A(z) - \sup A(x) < q - \inf A(x) = \tau(x). \tag{7}$$

Moreover, due to (4) and (6), we get

$$\big[A(x) + s\big] \cap A(y) \neq \emptyset$$

and

$$A(y) \cap \big[A(z) - t\big] \neq \emptyset.$$

Hence, according to the inequality (7) and the condition (H1),

$$\big[A(x) + s\big] \cap \big[A(z) - t\big] \neq \emptyset.$$

On the other hand, by (3) and (5), we obtain

$$\sup A(x) + s < \inf A(z) - t,$$

whence

$$\big[A(x) + s\big] \cap \big[A(z) - t\big] = \emptyset.$$

This contradiction completes the proof. $\square$

Define the following sets:

$$\begin{aligned}
\mathcal{L} &:= \big\{A(x)\colon x \in X, \inf A(x) = -\infty \text{ and } \infty \neq q \notin A(x)\big\},\\
\mathcal{S} &:= \big\{A(x)\colon x \in X, \operatorname{card} A(x) = 1 \text{ and } q \notin A(x)\big\},\\
\mathcal{P}_{-\infty} &:= \big\{A(x)\colon x \in X, \inf A(x) = -\infty \text{ and } q \in A(x)\big\},\\
\mathcal{P} &:= \big\{A(x)\colon x \in X, \inf A(x) \in A(x) \text{ and } q \in A(x)\big\},\\
\mathcal{L}_{-\infty} &:= \big\{A(x)\colon x \in X, \inf A(x) = -\infty \text{ and } \sup A(x) < q = \infty\big\},\\
\mathcal{P}_{\infty} &:= \big\{A(x)\colon x \in X, \inf A(x) > -\infty \text{ and } \sup A(x) = q = \infty\big\},\\
\mathcal{R} &:= \big\{A(x)\colon x \in X, \inf A(x) = -\infty \text{ and } \sup A(x) = q = \infty\big\}.
\end{aligned}$$

Assume that $\mathcal{A}$ and $\mathcal{B}$ are arbitrary families of subsets of $\mathbb{R}$. We will write $\mathcal{A} \preceq \mathcal{B}$ if

$$\sup A \leqslant \inf B$$

for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Let $F : (0, \infty) \times X \to 2^X$ be given by the following formula

$$F(t, x) := A^{-}(A(x) + \min\{t, q - \inf A(x)\}), \tag{A}$$

where

$$A^{-}(V) := \{x \in X\colon\ A(x) \cap V \neq \emptyset\}$$

for every $V \subset \mathbb{R}$.

In [5, Lemma 3] we proved the following fact.

Fact 1. (see [5, Lemma 3]) *Let $F : (0, \infty) \times X \to 2^X$ be given by* (A) *and let $t \in (0, \infty)$ and $x \in X$. If $t < \tau(x)$ then*

$$F(t, x) = A^{-}\big(A(x) + t\big) \neq \emptyset$$

and if $t \geqslant \tau(x)$ then

$$F(t, x) = \begin{cases} A^{-}(\{q\}), & \textit{if } q \in S \textit{ and } \inf A(x) \in A(x);\\ \emptyset & \textit{otherwise.}\end{cases}$$

Fact 2. (see [6, Theorem 3]) *Let $F : (0, \infty) \times X \to 2^X$ be given by* (A)*. If A is single-valued then F is an iteration semigroup.*

Now we present three theorems which was proved in [8]. We will use them in the proof of the main result of this paper.

Fact 3. (see [8, Theorem 2]) *Let* $F : (0, \infty) \times X \to 2^X$ *be given by* (A)*. Assume that* $q = \infty$*. Then* F *is an iteration semigroup if and only if* (H1) *is satisfied.*

Fact 4. (see [8, Theorem 5]) *Let* $F : (0, \infty) \times X \to 2^X$ *be given by* (A)*. Assume that* $q \notin S$ *and* $q \neq \infty$*. Then* F *is an iteration semigroup if and only if* (H1) *is satisfied and*

$$A(x) \in \mathcal{L} \cup \mathcal{S} \quad \textit{for} \;\; x \in X$$

and $\mathcal{L} \preceq \mathcal{S}$*.*

Fact 5. (see [8, Theorem 4]) *Let* $F : (0, \infty) \times X \to 2^X$ *be given by* (A)*. Assume that* $q \in S$*. Then* F *is an iteration semigroup if and only if the condition* (H1) *and all the following conditions hold:*

(a) $A(x) \in \mathcal{L} \cup \mathcal{S} \cup \mathcal{P}_{-\infty} \cup \mathcal{P}$ *for every* $x \in X$*;*
(b) $\mathcal{L} \preceq \mathcal{S} \cup \mathcal{P}$*;*
(c) $\mathcal{S} \preceq \mathcal{P}$*;*
(d) $\mathcal{S} = \emptyset$ *or* $\mathcal{P}_{-\infty} = \emptyset$*;*
(e) $\mathcal{L} = \emptyset$ *or* $\mathcal{P}_{-\infty} = \emptyset$ *or* $\mathcal{P} = \emptyset$*;*
(f) *for every* $x, y \in X$ *if* $A(x) \in \mathcal{P}_{-\infty} \cup \mathcal{P}$*,* $A(y) \in \mathcal{P}$*,* $\inf A(x) < \inf A(y)$ *and there exists* $s \in (0, \inf A(y) - \inf A(x))$ *satisfying* (1)*, then for every* $P \in \mathcal{P} \cup \mathcal{P}_{-\infty}$ *and* $t \in \big[\tau(y), \tau(x) - s\big)$ *the condition*

$$[A(x) + s + t] \cap P \neq \emptyset$$

holds;
(g) *for every* $x, y \in X$ *if* $A(x) \in \mathcal{L}$*,* $A(y) \in \mathcal{S} \cup \mathcal{P}$ *and* $s \in (0, \infty)$ *satisfies* (1)*, then the condition*

$$\big[A(x) + s + t\big] \cap P \neq \emptyset$$

holds for every $P \in \mathcal{P}$ *and* $t \geqslant \tau(y)$*.*

Since in this paper we are interested in multifunctions F given by (A) which are generated by interval-valued functions A, below we prove two properties of the multifunction A under this assumption.

Remark 4. Assume that the values of A are intervals and the condition (H1) holds. Let $x, y, z \in X$.

(i) If

$$\sup A(x) \leqslant \inf A(y) \leqslant \inf A(z), \tag{8}$$

then card $A(y) = 1$ or $\inf A(y) = \inf A(z)$.

(ii) If $\sup A(x) \leqslant \sup A(y) \leqslant \inf A(z)$, then card $A(y) = 1$ or $\sup A(x) = \sup A(y)$.

Proof. (i) Assume (8) and suppose that card $A(y) > 1$ and

$$\inf A(y) < \inf A(z). \tag{9}$$

Then, by Remark 3, we have

$$\inf A(z) < \sup A(y). \tag{10}$$

Let

$$t := \frac{\inf A(z) - \inf A(y)}{4},$$

and

$$s := \inf A(y) - \sup A(x) + t.$$

Of course $s, t \in (0, \infty)$ and

$$\inf A(y) < \inf A(y) + t = \sup A(x) + s.$$

Due to (9) we obtain

$$\begin{aligned} \inf A(z) - t = \inf A(z) - 2t + t = \frac{\inf A(z) + \inf A(y)}{2} + t > \\ > \inf A(y) + t = \sup A(x) + s. \end{aligned} \tag{11}$$

Hence, by (10),

$$\inf A(y) < \sup A(x) + s < \inf A(z) - t < \sup A(y).$$

Therefore, since the values of A are intervals, we get

$$\big[A(x) + s\big] \cap A(y) \neq \emptyset \quad \text{and} \quad [A(z) - t] \cap A(y) \neq \emptyset.$$

On the other hand, by (11)

$$[A(x) + s] \cap [A(z) - t] = \emptyset,$$

which, by the following inequality

$$\begin{aligned} s + t = \frac{\inf A(y) + \inf A(z) - 2\sup A(x)}{2} < \inf A(z) - \sup A(x) < \\ < q - \inf A(x) = \tau(x), \end{aligned}$$

contradicts condition (H1) and completes the proof of (i).

The proof of (ii) is similar. It is enough to take

$$s := \frac{\sup A(y) - \sup A(x)}{4} \quad \text{and} \quad t := \inf A(z) - \sup A(y) + s.$$

□

Notice that if A is interval-valued then for every $x \in X$ we get

$$
\begin{aligned}
A(x) \in \mathcal{L} &\text{ iff } A(x) = (-\infty, a_x) \text{ for some } a_x \leqslant q < \infty \text{ or } A(x) = (-\infty, a_x] \\
&\quad \text{for some } a_x < q < \infty, \\
A(x) \in \mathcal{S} &\text{ iff } A(x) = \{a_x\} \text{ for some } a_x \neq q, \\
A(x) \in \mathcal{P}_{-\infty} &\text{ iff } A(x) = (-\infty, q], \\
A(x) \in \mathcal{P} &\text{ iff } A(x) = [a_x, q] \text{ for some } a_x \leqslant q, \\
A(x) \in \mathcal{L}_{-\infty} &\text{ iff } A(x) = (-\infty, a_x) \text{ for some } a_x < q = \infty \text{ or } A(x) = (-\infty, a_x] \\
&\quad \text{for some } a_x < q = \infty, \\
A(x) \in \mathcal{P}_{\infty} &\text{ iff } A(x) = (a_x, \infty) \text{ for some } a_x \in \mathbb{R} \text{ or } A(x) = [a_x, \infty) \text{ for some} \\
&\quad a_x \in \mathbb{R}, \\
A(x) \in \mathcal{R} &\text{ iff } A(x) = \mathbb{R}.
\end{aligned}
$$

Lemma 1. *Assume that the values of A are intervals,* (H1) *holds and $q = \infty$. Then all the following conditions are satisfied*

(i) $A(x) \in \mathcal{L}_{-\infty} \cup \mathcal{S} \cup \mathcal{P}_{\infty} \cup \mathcal{R}$ *for every* $x \in X$;
(ii) $\mathcal{L}_{-\infty} \preceq \mathcal{S} \cup \mathcal{P}_{\infty}$;
(iii) $\mathcal{S} \preceq \mathcal{P}_{\infty}$;
(iv) $\mathcal{S} = \emptyset$ *or* $\mathcal{R} = \emptyset$;
(v) $\mathcal{L}_{-\infty} = \emptyset$ *or* $\mathcal{R} = \emptyset$ *or* $\mathcal{P}_{\infty} = \emptyset$.

Proof. Notice that $\tau(x) = \infty$ for every $x \in X$. The condition (i) follows immediately from Remark 2.

Pass to the proof of (ii). Suppose that there exist $x, y, z \in X$ such that $A(x) \in \mathcal{S} \cup \mathcal{P}_{\infty}$, $A(y) \in \mathcal{L}_{-\infty}$ and $\sup A(y) > \inf A(x)$. Take

$$s := \frac{\sup A(y) - \inf A(x)}{2},$$

and $t \in (0, \infty)$, $t > s$. Of course $s, t \in (0, \infty)$ and $s + t < \tau(x)$. Since the values of A are intervals, notice that

$$\big[A(x) + s\big] \cap A(y) \neq \emptyset \quad \text{and} \quad [A(y) - t] \cap A(y) \neq \emptyset,$$

but $[A(x) + s + t] \cap A(y) = \emptyset$, which contradicts (H1).

Now we prove (iii) and (iv). Suppose that there exist $x, y \in X$ such that

$$A(x) \in \mathcal{S} \quad \text{and} \quad A(y) \in \mathcal{P}_{\infty} \quad \text{and} \quad \inf A(y) < \sup A(x)$$

or

$$A(x) \in \mathcal{S} \quad \text{and} \quad A(y) \in \mathcal{R}.$$

Let $s \in (0, \infty)$ and $t \in \big(0, \sup A(x) - \inf A(y)\big)$. By our assumptions

$$\big[A(x) + s\big] \cap A(y) \neq \emptyset \quad \text{and} \quad [A(x) - t] \cap A(y) \neq \emptyset.$$

On the other hand

$$[A(x)+s]\cap[A(x)-t]=\emptyset,$$

which contradicts (H1).

To prove (v) suppose that there exist $x,y,z\in X$ such that $A(z)\in\mathcal{L}_{-\infty}$, $A(y)\in\mathcal{R}$ and $A(x)\in\mathcal{P}_{\infty}$. By (ii) we get

$$\sup A(z)\leqslant\inf A(x). \tag{12}$$

For every $s,t\in(0,\infty)$ we obtain

$$\big[A(x)+s\big]\cap A(y)\neq\emptyset\quad\text{and}\quad\big[A(z)-t\big]\cap A(y)\neq\emptyset.$$

Due to (12)

$$\big[A(x)+s\big]\cap\big[A(z)-t\big]=\emptyset,$$

which contradicts (H1) and completes the proof of (v). □

2. Main result

Let $\mathcal{A},\mathcal{B}\subset 2^{\mathbb{R}}$. We will say that A *has values of type* $\mathcal{A}$, if $A(x)\in\mathcal{A}$ for every $x\in X$.

We will say that A *has values of type* $\mathcal{AB}$, if $A(x)\in\mathcal{A}\cup\mathcal{B}$ for every $x\in X$ and $\mathcal{A}\neq\emptyset$ and $\mathcal{B}\neq\emptyset$. Similarly for three classes of sets.

We will say that $\mathcal{A}$ *has property* W and write $W(\mathcal{A})$, if

$$\operatorname{int}P=\operatorname{int}R,\quad\text{for every }\ P,R\in\mathcal{A},$$

(where $\operatorname{int}P$ denotes the interior of the set P).

Now we present the main result of this paper.

Theorem. *Assume that the values of A are intervals and $F:(0,\infty)\times X\to 2^X$ is given by* (A)*. Then F is an iteration semigroup if and only if one of the following conditions holds:*

(i) *A has values of type $\mathcal{P}$;*
(ii) *A has values of type $\mathcal{P}_{-\infty}$;*
(iii) *A has values of type $\mathcal{PP}_{-\infty}$;*
(iv) *A has values of type $\mathcal{SP}$ and*

$$\mathcal{S}\preceq\mathcal{P}\quad\textit{and}\quad\operatorname{card}\mathcal{P}=1;$$

(v) *A has values of type $\mathcal{LP}$ and*

$$\mathcal{L}\preceq\mathcal{P}\quad\textit{and}\quad W(\mathcal{L})\quad\textit{and}\quad\operatorname{card}\mathcal{P}=1;$$

(vi) *A has values of type $\mathcal{LP}_{-\infty}$;*
(vii) *A has values of type $\mathcal{LSP}$ and*

$$\mathcal{L}\preceq\mathcal{S}\preceq\mathcal{P}\quad\textit{and}\quad W(\mathcal{L})\quad\textit{and}\quad\operatorname{card}\mathcal{P}=1;$$

(viii) *A has values of type $\mathcal{S}$;*

(ix) *A has values of type* $\mathcal{L}$*;*
(x) *A has values of type* $\mathcal{LS}$ *and*

$$\mathcal{L} \preceq \mathcal{S} \quad \textit{and} \quad W(\mathcal{L});$$

(xi) *A has values of type* $\mathcal{P}_\infty$*;*
(xii) *A has values of type* $\mathcal{L}_{-\infty}$*;*
(xiii) *A has values of type* $\mathcal{R}$*;*
(xiv) *A has values of type* $\mathcal{L}_{-\infty}\mathcal{P}_\infty$ *and*

$$\mathcal{L}_{-\infty} \preceq \mathcal{P}_\infty \quad \textit{and} \quad W(\mathcal{L}_{-\infty}) \quad \textit{and} \quad W(\mathcal{P}_\infty);$$

(xv) *A has values of type* $\mathcal{SP}_\infty$ *and*

$$\mathcal{S} \preceq \mathcal{P}_\infty \quad \textit{and} \quad W(\mathcal{P}_\infty);$$

(xvi) *A has values of type* $\mathcal{L}_{-\infty}\mathcal{R}$*;*
(xvii) *A has values of type* $\mathcal{RP}_\infty$*;*
(xviii) *A has values of type* $\mathcal{L}_{-\infty}\mathcal{S}$ *and*

$$\mathcal{L}_{-\infty} \preceq \mathcal{S} \quad \textit{and} \quad W(\mathcal{L}_{-\infty});$$

(xix) *A has values of type* $\mathcal{L}_{-\infty}\mathcal{SP}_\infty$ *and*

$$\mathcal{L}_{-\infty} \preceq \mathcal{S} \preceq \mathcal{P}_\infty \quad \textit{and} \quad W(\mathcal{L}_{-\infty}) \quad \textit{and} \quad W(\mathcal{P}_\infty).$$

Proof. At first assume that F is an iteration semigroup. Of course, by Facts 3, 4 and 5, condition (H1) is satisfied. We show that one of the conditions (i)–(xix) holds.

First we prove that the multifunction A can have only the values mentioned in conditions (i)–(xix).

Consider the case $q \in \mathcal{S}$. Then, by Fact 5, F has the values in the set $\mathcal{L} \cup \mathcal{S} \cup \mathcal{P}_{-\infty} \cup \mathcal{P}$. Notice that if all of the values of A belong to the same class, then it can be $\mathcal{P}_{-\infty}$ [cf. (ii)] or $\mathcal{P}$ [cf. (i)] (because $q \in S$). Assume that A has values in exactly two classes of sets. Then we have 6 possibilities. Since $q \in S$ it cannot be $\mathcal{LS}$, and, by the condition (d) of Fact 5, it cannot be $\mathcal{SP}_{-\infty}$. Thus we obtain the types from (iii)–(vi). Notice that if A has values in exactly three classes of sets, then, by Fact 5(d) and (e), it has to be the type $\mathcal{LSP}$ [cf. (vii)]. Due to the condition (e) of Fact 5, there does not exist $x \in X$ such that $A(x) \in \mathcal{LSP}_{-\infty}\mathcal{P}$.

Now pass to the case when $q \notin S$ and $q \neq \infty$. Therefore, according to Fact 4, the values of A can be of the types: $\mathcal{L}$, $\mathcal{S}$ or $\mathcal{LS}$ (cf. (viii)-(x)).

Assume that $q = \infty$. Then, by Lemma 1, for every $x \in X$

$$A(x) \in \mathcal{L}_{-\infty} \cup \mathcal{S} \cup \mathcal{P}_\infty \cup \mathcal{R}.$$

If A has only one kind of values then each of the types: $\mathcal{L}_{-\infty}$, $\mathcal{S}$, $\mathcal{P}_\infty$, $\mathcal{R}$ is possible [cf. (viii), (xi)–(xiii)]. If A has values in exactly two classes of sets then by Lemma 1(iv), it cannot be $\mathcal{SR}$. Thus the conditions (xiv)–(xviii) describe all the types of the values in this case. Observe that if A has values in exactly three classes of sets, then, by Lemma 1[(iv) and (v)], it has to be the type $\mathcal{L}_{-\infty}\mathcal{SP}_{\infty}$ [cf. (xix)] and according to the condition (v) of Lemma 1, the multifunction A cannot have values of the type $\mathcal{L}_{-\infty}\mathcal{SP}_{\infty}\mathcal{R}$.

The relations "$\preceq$" between the classes of sets follow immediately from Fact 5 [in the cases (iv), (v) and (vii)], from Fact 4 [in the case (x)] and from Lemma 1 [in the cases (xiv)–(xv) and (xviii)–(xix)].

The conditions relating to the cardinality of sets and property W follow immediately from Remark 4.

Now pass to the proof of the contrary implication. At first notice that if (viii) holds then A is single-valued and, by Fact 2, F is an iteration semigroup. In the cases (i)–(iii), (vi), (ix), (xi)–(xiii) and (xvi)–(xvii) the condition

$$[A(x) + s] \cap A(z) \neq \emptyset$$

is satisfied for all $x, z \in X$ and $s \in \big(0, \tau(x)\big] \cap \mathbb{R}$. Hence (H1) holds. Moreover in this cases we have $F \equiv X$, so the multifunction A generates an iteration semigroup F. It easy to observe that also in other cases condition (H1) is satisfied. Of course each of the conditions (xiv)–(xv) and (xviii)–(xix) defines the multifunction A for which $q = \infty$. Therefore by Fact 3 in all these cases F is an iteration semigroup. In the cases (iv), (v) and (vii) we obtain that $q \in S$ and the conditions (a)–(g) of Fact 5 hold, thus F is an iteration semigroup. Now assume (x). Then $q \notin S$ and $q \neq \infty$, so due to Fact 4 we obtain that F is an iteration semigroup. $\square$

As follows from the above proof if A satisfies one of conditions: (i)–(iii), (vi), (ix), (xi)–(xiii), (xvi) or (xvii) of the Theorem, then $F \equiv X$.

Since in our paper we always assume that A satisfies condition (H), we can ask what it means if one of conditions (i)–(xix) of the Theorem holds. Notice that for an arbitrary interval-valued function A we obtain:

- if A satisfies one of the conditions: (i)–(iii), (vi), (ix), (xi)–(xiii), (xvi), (xvii), then S is an interval, thus by Remark 1 condition (H) holds;
- if at least one of the values of A is a singleton [see (iv), (vii), (viii), (x), (xv), (xviii), (xix)] then by Remark 1 we get: (H) holds if and only if S is an interval;
- if A satisfies (v) then it is easy to see that: (H) holds if and only if S is an interval;
- if A satisfies (xiv) then we can notice that: if at least one of values of A is a closed set, then (H) holds if and only if S is an interval, i.e. $S = \mathbb{R}$; otherwise: (H) holds if and only if card $(\mathbb{R} \setminus S) \leqslant 1$.

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References

[1] Guzik, G., Jarczyk, W., Matkowski, J.: Cocycles of continuous iteration semigroups. Bull. Polish Acad. Sci. Math. **51**, 187–197 (2003)

[2] Kuczma, M.: Functional equations in a single variable. In: Monografie Mat. vol. 46. PWN, Warszawa (1968)

[3] Kuczma, M., Choczewski, B., Ger, R.: Iterative functional equations. In: Encyclopedia of Math. and its Applications, vol. 32. Cambridge University Press, Cambridge (1990)

[4] Łydzińska, G.: Iteration families for which expansion implies collapse. Aeqationes Math. **70**, 247–253 (2005)

[5] Łydzińska, G.: On collapsing iteration semigroups of set-valued functions. Publ. Math. Debr. **64**, 285–298 (2004)

[6] Łydzińska, G.: On expanding iteration semigroups of set-valued functions. Math. Pannon. **15**, 55–64 (2004)

[7] Łydzińska, G.: On lower semicontinuity of some set-valued iteration semigroups. Nonlinear Anal. **71**, 5644–5654 (2009)

[8] Łydzińska, G.: The characterization of some set-valued iteration semigroups. Aequationes Math. (2014). doi:10.1007/s00010-014-0270-x

[9] Olko, J.: Selections of an iteration semigroup of linear set-valued functions. Aequationes Math. **56**, 157–168 (1998)

[10] Olko, J.: On indexed families of multifunctions generated by families of functions. Aequationes Math. **88**, 125–135 (2014)

[11] Smajdor A.: Iterations of multi-valued functions. Prace Naukowe Uniwersytetu Śląskiego w Katowicach 759 (1985)

[12] Smajdor, A.: On concave iteration semigroups of linear set-valued functions. Aequationes Math. **75**, 149–162 (2008)

[13] Targonski, Gy.: Topics in Iteration Theory. Vandenhoeck and Ruprecht, Göttingen (1981)

[14] Zdun, M.C.: Continuous and differentiable iteration semigroups. Prace Naukowe Uniwersytetu Śląskiego w Katowicach 308 (1979)

[15] Zdun, M.C.: On set-valued iteration groups generated by commuting functions. J. Math. Anal. Appl. **398**, 638–648 (2013)

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